

# About one conjecture of twin domination number of oriented graphs

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## Abstract

Let  $D = (V, A)$  be a digraph. A subset  $S$  of  $V$  is called a twin dominating set of  $D$  if for every vertex  $v \in V - S$ , there exists vertices  $u_1, u_2 \in S$  such that  $(v, u_1)$  and  $(u_2, v)$  are arcs in  $D$ . The minimum cardinality of a twin dominating set in  $D$  is called the twin domination number of  $D$  and is denoted by  $\gamma^*(D)$ .

In [2], is defined the concept of upper orientable twin domination number of a graph  $G$ ,  $DOM^*(G) = \max\{\gamma^*(D) | D \text{ is an orientation of } G\}$ . In [1], it is conjectured that for the complete graph  $K_n$  with  $n \geq 8$ ,  $DOM^*(K_n) = \lceil \frac{n+1}{2} \rceil$ . In this work we prove that  $DOM^*(K_n) \leq \lceil \frac{n}{2} \rceil$  for all even number  $n \geq 8$ .

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## 1 Introduction

Let  $D = (V, A)$  be a digraph. For any vertex  $v \in V$ , the sets  $I(v) = \{u | (u, v) \in A\}$  and  $O(v) = \{u | (v, u) \in A\}$  are called the inset and outset of  $v$ . The indegree and outdegree of  $v$  are defined by  $id(v) = |I(v)|$  and  $od(v) = |O(v)|$ .

**Definition 1** Let  $D = (V, A)$  be a digraph. A subset  $S$  of  $V$  is called a *twin dominating set* of  $D$  if for every vertex  $v \in V - S$ , there exists vertices  $u_1, u_2 \in S$  ( $u_1, u_2$  may coincide) such that  $(v, u_1)$  and  $(u_2, v)$  are arcs in  $D$ . The minimum cardinality of a twin dominating set in  $D$  is called the *twin domination number* of  $D$  and is denoted by  $\gamma^*(D)$ .

For different orientations  $D_1$  and  $D_2$  of a graph  $G$ , it is possible to have  $\gamma^*(D_1) \neq \gamma^*(D_2)$ . In [2], the authors defined the concept of upper orientable twin domination number of a graph  $G$ ,

$$DOM^*(G) = \max\{\gamma^*(D) | D \text{ is an orientation of } G\}.$$

In [1], it is conjectured that for  $n \geq 8$  and the complete graph  $K_n$ ,  $DOM^*(K_n) = \lceil \frac{n+1}{2} \rceil$ . We will prove in section 3 that for  $n = 8$ ,  $DOM^*(K_8) = 4$ .

In section 4, we prove that  $DOM^*(K_n) \leq \lceil \frac{n}{2} \rceil$  for all even number  $n \geq 8$ .

## 2 Previous result and lemmas

The following observation is a consequence of the results given in [1].

**Observation 1** Let  $T$  be a tournament (one orientation of the complete graph) of order  $n \geq 3$ . From Theorem 2.6 [1], if  $T$  contains at least one vertex  $u \in V(T)$  such that  $id(u) = 0$  or  $od(u) = 0$ , then  $\gamma^*(T) \leq \lceil \log_2(n-1) + 1 \rceil$ . In the case of  $n \geq 8$ ,  $\gamma^*(T) \leq \lceil \frac{n}{2} \rceil$ .

**Theorem 2** Let  $T$  be an orientation of  $K_8$ . If there exists  $v \in V(T)$  such that  $id(v) = 2$  or  $od(v) = 2$ , then  $\gamma^*(T) \leq 4$ .

**Proof.** Suppose there exists a vertex  $v \in V(T)$  such that  $id(v) = 2$ ,  $I(v) = \{i, i'\}$  and  $O(v) = \{o_1, o'_1, o_2, o'_2, z\}$ . Without loss of generality we can assume that the arcs  $(i, i'), (o'_1, o_1), (o'_2, o_2) \in A(T)$ .

1. If the arcs  $(z, i), (z, o_1)$  or  $(z, o_2)$  are in  $A(T)$ , then  $S = \{v, i, o_1, o_2\}$  is a twin dominating set of  $T$ . So we can assume that the arcs  $(i, z), (o_1, z), (o_2, z) \in A(T)$ . See Figure 1.
2. If the arc  $(o'_1, o'_2) \in A(T)$ , then the set  $S = \{v, i, z, o'_2\}$  is a twin dominating set of  $T$ . If the arc  $(o'_2, o'_1) \in A(T)$ , then the set  $S = \{v, i, z, o'_1\}$  is a twin dominating set of  $T$ .

Therefore, if  $id(v) = 2$ ,  $\gamma^*(T) \leq 4$ . The case  $od(v) = 2$ , is symmetric. ■

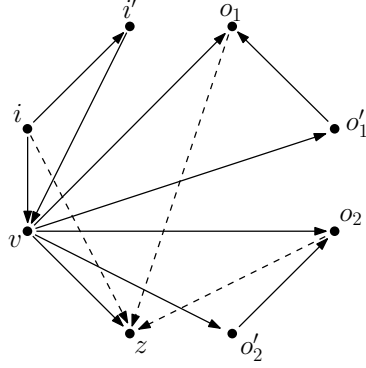


Figure 1: Illustration of Step 1 from Theorem 2 for the tournament  $T$  of order 8, which has a vertex  $v$  of  $id(v) = 2$ ; *dashed arrows* denote new added edges in Step 1.

**Theorem 3** *Let  $T$  be an orientation of  $K_8$ . If there exists  $v \in V(T)$  such that  $id(v) = 1$  or  $od(v) = 1$ , then  $\gamma^*(T) \leq 4$ .*

**Proof.** Let  $v \in V(T)$  such that  $id(v) = 1$  with  $I(v) = \{z\}$  and  $O(v) = \{o_1, o_1', o_2, o_2', o_3, o_3'\}$ . Without loss of generality we can assume that the arcs  $(o_1', o_1), (o_2', o_2), (o_3', o_3) \in A(T)$ .

1. If  $(o_1, z), (o_2, z)$  or  $(o_3, z)$  are in  $A(T)$ , then  $S = \{v, o_1, o_2, o_3\}$  is a twin dominating set of  $T$ . So  $\{v, o_1, o_2, o_3\} \subset O(z)$ .
2. By the Observation 1,  $id(z) \neq 0$ , in other case,  $\gamma^*(T) \leq 4$ . Assume, without lost of generality, that  $(o_3', z) \in A(T)$ . If one of the arcs  $(o_3, o_1)$  or  $(o_3, o_2)$  are in  $T$ , then  $S = \{v, z, o_1, o_2\}$  is a twin dominating set of  $T$ . So let us assume that  $(o_1, o_3), (o_2, o_3) \in A(T)$ , see Figure 2.
3. If  $(o_3, o_1')$  and  $(o_3, o_2')$  are arcs in  $T$ , then  $od(o_3) = 2$  and from Theorem 2,  $\gamma^*(T) \leq 4$ . On the other hand, if  $(o_1', o_3)$  (resp.  $(o_2', o_3)$ ) is an arc in  $T$ , then  $S = \{v, z, o_2, o_3\}$  (resp.  $S = \{v, z, o_1, o_3\}$ ) is a  $\gamma^*$ -set of  $T$ .

The case  $od(v) = 1$  is symmetric. ■

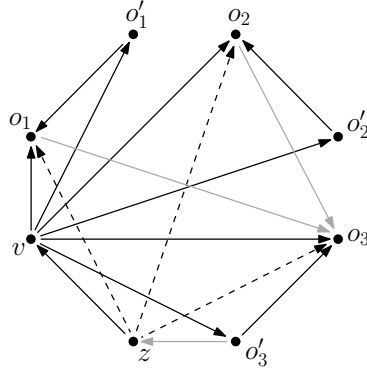


Figure 2: Illustration of Step 1-2 from Theorem 3 for the tournament  $T$  of order 8, which has a vertex  $v$  of  $id(v) = 1$ ; *dashed arrows* denote new added edges in Step 1 and *gray arrows* denote new added edges in Step 2.

**Observation 4** Denote by  $T_n^6$  the tournament of order  $n + 6$  of the Figure 3. In [1] is proved that  $\gamma^*(T_0^6) = \gamma^*(T_1^6) = 4$ . So for any  $n \geq 0$ ,  $\gamma^*(T_n^6) = 4$  and  $4 \leq DOM^*(K_n)$ .

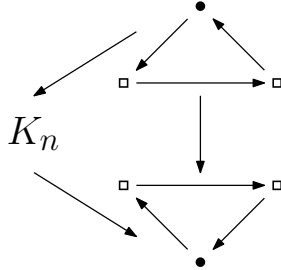


Figure 3: Illustration for Observation 4; *boxes* denote the  $\gamma^*$ -set of  $T_n^6$  of size 4, for  $n \geq 0$ .

**Theorem 5** The upper orientable twin domination number of  $K_8$ ,  $DOM^*(K_8) = 4$ .

**Proof.**

Let  $T$  be an orientation of  $K_8$ . By the previous observation, if there exist  $v \in V(T)$  such that  $id(v) \in \{0, 1, 2, 5, 6\}$  or  $od(v) \in \{0, 1, 2, 5, 6\}$ , then  $\gamma^*(T) \leq 4$ . So, we can suppose that for every  $v \in V(T)$ ,  $id(v) = 3$  or  $id(v) = 4$ .

Let  $v \in V(T)$  such that  $O(v) = \{o_1, o'_1, o_2, o'_2\}$  and  $I(v) = \{i, i', z\}$ . Without lost of generality, we can suppose that the arcs  $(i, i'), (o'_1, o_1), (o'_2, o_2) \in A(T)$ .

If one of the arcs  $(i, z), (o_1, z)$  or  $(o_2, z)$  are in  $A(T)$ , then  $S = \{v, i, o_1, o_2\}$  is a twin dominating set of  $T$ . So, we can suppose that  $O(z) = \{v, i, o_1, o_2\}$  and  $I(z) = \{i', o'_1, o'_2\}$ .

1. If  $(o_1, o_2) \in A(T)$ , then  $S = \{v, z, i, o_2\}$  is a twin dominating set of  $T$ .
2. If  $(o_2, o_1) \in A(T)$ , then  $S = \{v, z, i, o_1\}$  is a twin dominating set of  $T$ .

The case of  $v \in V(T)$  such that  $id(4) = 4$  is symmetric. Therefore,  $DOM^*(K_8) \leq 4$ . By the previous observation, we can conclude that  $DOM^*(K_8) = 4$ . ■

### 3 Main result

In this section we proof that for any even number  $n \geq 8$ , the conjecture given in [1] is false.

**Theorem 6** *For every even natural number  $n \geq 8$ , the upper orientable twin domination number of  $K_n$ ,  $DOM^*(K_n) \leq \frac{n}{2}$ .*

**Proof.** Let  $n = 2k$ ,  $k \geq 4$ . By use induction on  $k$ .

If  $k = 4$ , the theorem holds by Theorem 5.

Suppose the theorem is true for any tournament  $T_1$  of order  $2k$ . Let  $T_2$  to be a tournament of order  $2k + 2$ . Consider  $v_1, v_2 \in V(T_2)$ ,  $(v_1, v_2) \in A(T_2)$  and  $T_1$  the subtournament of  $T_2$  induced by  $V(T_1) = V(T_2) - \{v_1, v_2\}$ .

By our hypothesis induction, there exist a  $\gamma^*$ -set  $S_1$  of  $T_1$ , such that  $|S_1| \leq k$ . If  $|S_1| < k$ , then  $S_2 = S_1 \cup \{v_1, v_2\}$  is a twin dominating set of  $T_2$  with  $|S_2| \leq k + 1$ . So, suppose  $|S_1| = k$ .

1. If for one vertex  $v \in S_1$ ,  $(v, v_1) \in A(T_2)$  or  $(v_2, v) \in A(T_2)$ , then the set  $S_2 = S_1 \cup \{v_2\}$  or  $S_2 = S_1 \cup \{v_1\}$ , respectively, is a twin dominating set of  $T_2$  with cardinality  $|S_2| = k + 1$ . Therefore we can assume that  $v_1$  is a source and  $v_2$  a sink with respect to the set  $S_1$ . Notice that  $\{v_1, v_2\}$  is a twin dominating set of  $S_1$ .

2. If one of the vertices in  $T_2$  is source or sink the proof is finished according to Observation 1. So there exists  $o_1, o_2 \in V(T_2)$  such that the arcs  $(o_1, v_1), (v_2, o_2) \in A(T_2)$ . If  $o_1 = o_2$ ,  $v_1 v_2 o_2$  or  $o_1 v_1 v_2$  is an oriented cycle in  $T_2$ , then  $S_2 = V(T_2) \setminus \{S_1 \cup \{o_i\}\}$  with  $i \in \{1, 2\}$  is a twin dominating set of  $T_2$  with  $|S_2| = k + 1$ . Thus, the arcs  $(o_1, v_2), (v_1, o_2) \in A(T_2)$ . It is clear, that the set  $S_2 = V(T_2) \setminus S_1$  is a twin dominating set of  $T_2$  with cardinality  $|S_2| = k + 2$ . We will prove that  $S_2$  it is not minimum. Let  $v \in S_2 \setminus \{v_1, v_2, o_1, o_2\}$ .

- (a) If the arc  $(o_2, o_1) \in A(T_2)$ , then  $S_2 - \{o_2\}$  is a twin dominating set of  $T_2$  with cardinality  $k + 1$ . So, we can suppose we have the arc  $(o_1, o_2) \in A(T_2)$ , see Figure 4.
- (b) If the arc  $(v, o_1)$  or  $(o_2, v)$  is in  $A(T_2)$ , then  $S_2 - \{o_1\}$  or  $S_2 - \{o_2\}$  are twin dominating set of  $T_2$ , respectively. So we can assume that in  $T_2$ , we have the arcs  $(o_1, v), (v, o_2)$ , which implies that  $S_2 - \{v\}$  is a twin dominating set of  $T_2$  with cardinality  $k + 1$ .

■

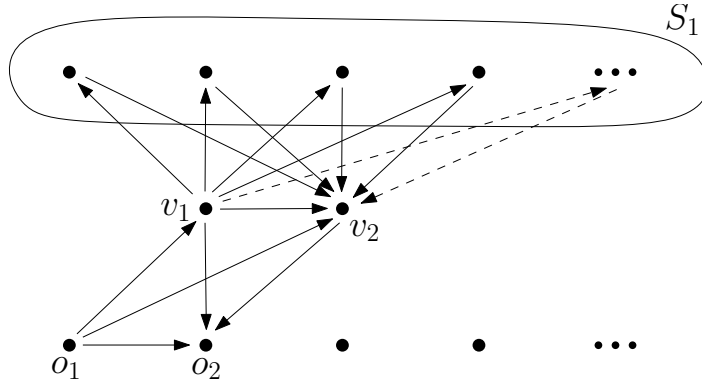


Figure 4: Tournament  $T_2$ ; *black dots* denote vertices from  $V(T_2)$ , where  $V(T_1) = V(T_2) \setminus \{v_1, v_2\}$  and  $S_1$  denotes the  $\gamma^*$ -set of  $T_1$ .

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